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STUDY OF REDUCTION FORMULAE FOR GENERALIZED $\widetilde{H}$ −FUNCTION AND ITS APPLICATIONS IN ARTIFICIAL INTELLIGENCE AND ROBOTICS IN LIFE SCIENCES

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Abstract: Reduction formulas are particularly useful when dealing with integrals of powers of basic functions like trigonometric functions logarithmic functions, and exponential functions. For example, if we have an integral involving a power of a trigonometric function, a reduction formula can help express it in terms of integrals involving lower powers, making the computation more manageable. Reduction formulas can handle integrals involving polynomials of any degree. This is significant because direct integration of high-degree polynomials can be complex and time-consuming. By applying reduction formulas, we can express the integral of a higher degree polynomial as a combination of integrals involving lower degree polynomials or simpler functions. Products of Reduction formulas are also applicable to integrals involving products of transcendental functions, such as the product of a trigonometric function and an exponential function. The use of reduction formulas allows breaking down the complex integral into simpler components, making it easier to evaluate. The versatility of reduction formulas extends to a wide range of mathematical expressions, enabling their application to various types of functions and integrals. The $\overline{H}$-function mentioned in our context seems to represent a generic function with parameters, and reduction formulas can be used to obtain specific results by specializing these parameters. Reduction formulas provide a systematic approach to solving higher order integrals. By expressing integrals in terms of lower order integrals, a step-by-step reduction process simplifies the computation. In this research paper we discuss Reduction Formulae for the Generalized $\widetilde{H}$-function.

Keywords: Trigonometric function, Generalized $\widetilde{H}$-function, Special Functions, Integrals, and Reduction Formulae, Artificial Intelligence, Robotics, Life Sciences.

2020 Mathematical Science Classification: 33-XX, 33CXX, 33-XX, 26-XX, 33Exx.

I. INTRODUCTION

Higher-order integrals can be expressed in terms of lower-order integrals using reduction formulas. They start with fundamental equations. This procedure makes it easier to compute integrals, which can be difficult to evaluate directly. By incorporating complicated functions like powers of basic functions, polynomials of any degree, and products of transcendental functions, you demonstrated how adaptable reduction formulae are to a wide range of mathematical expressions **[1]**. The $\widetilde{\boldsymbol{H}}$-function, a parametrizable function with mathematical flexibility, may provide surprising and unexpected results. The eight reduction equations developed for the $\widetilde{\boldsymbol{H}}$-function demonstrate the complexities of the $\widetilde{\boldsymbol{H}}$-function and the need for specific ways to manage its integrals. Mathematical study requires the development of reduction formulas and procedures for particular functions since they facilitate theoretical understanding and practical problem resolution. A solid understanding of the fundamentals of complex functions is essential in many scientific and technical fields. Usually, to find the reduction equations for a particular function, one must employ sophisticated mathematical techniques like recursive connections, special functions, or other methods **[2]**. For specific reduction equations for the generalized $\widetilde{\boldsymbol{H}}$-function, the original research articles, academic literature, or publications by mathematicians who have worked on this problem. Numerous fields in applied mathematics, engineering, and mathematical physics have uses for the robust and flexible $\widetilde{\boldsymbol{H}}$-function. This function, which is often updated to accommodate a larger range of parameters and inputs, makes assessing its integrals challenging due to its complicated nature **[3]**. To manage the difficulties associated with integrals using the generalized $\widetilde{\boldsymbol{H}}$-function, researchers have developed a number of reduction formulae. These reduction equations were developed in response to the requirement to simplify the computation of integrals involving the $\widetilde{\boldsymbol{H}}$-function and its derivatives **[4]**. Straight integration of the generalized $\widetilde{\boldsymbol{H}}$-function can be difficult since it encompasses a wide range of mathematical expressions. This work provides a series of reduction equations to better understand and use the generalized $\widetilde{\boldsymbol{H}}$-function. These equations can be used to recursively simplify integrals with the $\widetilde{\boldsymbol{H}}$-function. They are founded on fundamental principles and equations. The work exposes information on the function's mathematical structure by exploring the complexities of the $\widetilde{\boldsymbol{H}}$-function behavior under various parameterizations. May find detailed calculations and descriptions of integral transformations, mathematical analysis, and special functions in books or research publications**[5].**The reduction formula for the $\widetilde{\boldsymbol{H}}$-function , also known as the Fox $\widetilde{\boldsymbol{H}}$-function , is a mathematical relationship that allows the computation of one $\widetilde{\boldsymbol{H}}$-

function with parameters (a, b) in terms of another $\widetilde{\boldsymbol{H}}$-function with slightly different parameters**[6]**. The $\widetilde{\boldsymbol{H}}$-function is a generalization of the hyper geometric function and finds applications in various branches of mathematics and physics, particularly in problems involving complex integrals, special functions, and solutions to differential equations.

$$\widetilde{H}(a,b) = \frac{1}{2\pi i}\int_c \frac{\Gamma(a)}{\Gamma(b)} t^{-\alpha}(1-t)^{b-\alpha-1}e^t\, dt \qquad (1)$$

The H̃-function is a special function that arises in mathematical analysis, more precisely in the theory of fractional calculus. A Mellin-Barnes integral, which is frequently used to define the $\widetilde{\boldsymbol{H}}$-function, requires a contour integral in the complex plane **[7].** The H̃-function with parameters (a, b) may be found in its general form using a convergent series. The gamma function (Γ) and a contour integral along a suitable route in the complex plane are often needed for this. The gamma function is a widely used mathematical tool that may be used to extend the factorial function to complex numbers. The reduction formula for the $\overline{H}$-function involves expressing one $\widetilde{\boldsymbol{H}}$-function with parameters (a, b) in terms of other $\widetilde{\boldsymbol{H}}$-function with adjusted parameters **[8]**. This can be particularly useful for simplifying complex integrals and solving integral equations involving $\widetilde{\boldsymbol{H}}$-function. The reduction formula generally takes the following form

$$\widetilde{H}(a,b) = \text{combination of } \widetilde{\mathrm{H}}(\text{new a}, \text{new b}) + \text{other terms.} \qquad (2)$$

Context and particular characteristics determine the shape of the reduction formula. Mathematical approaches, the features of the integral representation, and the gamma function are utilized to generate reduction formulae for $\overline{H}$-function **[9]**. For the purpose of manipulating and evaluating -functions, reduction formulae are essential since they serve to simplify complex integrals and equations into forms that are easier to examine or solve. When dealing with unique functions in mathematics and its applications, they are a vital resource for practitioners and academics **[10]**. Applications for the reduction formula of the $\overline{H}$-function are many in the domains of mathematics, physics, engineering, and other disciplines. A few noteworthy uses are as follows:

1.1 Mathematical Physics: Reduction formulas for the $\widetilde{\boldsymbol{H}}$-function are commonly used in solving integral equations and differential equations arising in mathematical physics. They can simplify complex integrals that appear in the context of wave equations, heat equations, and other partial differential equations. This simplification often leads to analytical solutions or more tractable forms of equations. Reduction formulae for the $\widetilde{\boldsymbol{H}}$-function are essential tools in mathematical physics that are used to solve differential and integral problems **[11]**.In mathematical physics, integral equations and PDEs sometimes include complex mathematical statements that are difficult to solve analytically. The $\widetilde{\boldsymbol{H}}$-function reduction formulae offer a methodical way to decompose these intricate integrals into smaller, easier-to-manage parts. These reduction formulae achieve this by converting the original issue into a format that can be solved analytically, which makes it easier to derive solutions.

1.2 Probability and Statistics: $\widetilde{\boldsymbol{H}}$-function appears in various contexts in probability theory and statistics, such as in the computation of probability density functions and cumulative distribution functions for certain probability distributions. Reduction formulas can help simplify expressions involving these functions, making statistical calculations more manageable **[12]**. The $\widetilde{\boldsymbol{H}}$-function often occurs when the probability distribution has complicated expressions or when many distributions are convoluted. In these cases, reduction equations for the $\widetilde{\boldsymbol{H}}$-function are quite useful. These equations provide a systematic way to simplify the expressions required for calculating PDFs and CDFs, facilitating the analysis of the underlying probabilistic models. Reduction formulas can be especially handy for intricate operations like convolutions and products that include probability distributions. By using reduction equations to break down these processes into smaller parts, statisticians and probability may quickly handle complex expressions and provide analytical results for a variety of statistical measures.

1.3 Quantum Mechanics: In quantum mechanics, the $\widetilde{\boldsymbol{H}}$-function can appear when dealing with wave functions, energy eigen values, and scattering problems. Reduction formulas can help simplify integrals that arise in quantum mechanical calculations, aiding in the determination of physical quantities and solutions to Schrödinger's equation **[13]**.When dealing with complex mathematical expressions, convolutions, and probability distributions, the $\widetilde{\boldsymbol{H}}$-function is an essential tool in probability theory and statistics. Reduction equations for the $\widetilde{\boldsymbol{H}}$-function become essential when computing probability density functions (PDFs), cumulative distribution functions (PDFFs), or the convolution of many distributions. Reduction formulae are most useful when they can be used to systematically reduce the formulations needed to compute PDFs and CDFs. Reduction equations give statisticians and probability an organized way to handle convolutions and products using probability distributions, which enables them to deconstruct complicated processes into smaller parts **[14]**. Consequently, this makes it easier to analyze the underlying probabilistic models in a more effective and efficient manner.

1.4 Electromagnetic: The complex mathematical formulas describing electromagnetic interactions lead to the participation of $\widetilde{\boldsymbol{H}}$-function in these integrals. Reduction formulae are essential tools for solving the problems these intricate integrals provide. Convolutions, products, and other complex processes that can be difficult to handle directly are frequently involved in electromagnetic difficulties **[15]**. Researchers studying electromagnetic may systematically reduce these integrals by using reduction formulae for $\widetilde{\boldsymbol{H}}$-function, which offers a more manageable path to answers.

1.5 Engineering and Signal Processing: The $\widetilde{\boldsymbol{H}}$-function is a useful mathematical tool in engineering, especially in signal processing, control theory, and communication systems. In systems analysis and design, it is frequently used in conjunction with signal processing, filtering procedures, and dynamic processes **[16]**. For tackling the mathematical difficulties involved in complex system dynamics, filter design, and signal analysis, reduction formulae for the $\widetilde{\boldsymbol{H}}$-function are indispensable. Simplifying complex integrals and expressions that emerge during the modeling and analysis of engineering systems is made easier with the help of these reduction formulae. $\widetilde{\boldsymbol{H}}$-function has applications in various engineering fields, such as control theory, signal processing, and communication systems. Reduction formulas can be useful for solving problems related to system dynamics, filter design, and signal analysis **[17].** Reduction formulas are especially useful when determining the electromagnetic fields surrounding intricate structures, analyzing antenna radiation patterns, and comprehending scattering issues where electromagnetic waves interact with objects. In these situations, using reduction formulas facilitates the decomposition of the intricate mathematical expressions used in the computations, enabling more effective and perceptive analyses **[18]**. The reduced expressions produced by reduction formulas aid in the comprehension of the underlying electromagnetic phenomena. Through its use, researchers may get valuable insights about the properties of radiated waves, the dispersion of electromagnetic fields, and the interactions between electromagnetic waves and different materials **[19]**. In control theory, the $\widetilde{\boldsymbol{H}}$-function is useful for examining transfer functions, linear time-invariant systems, and the responses of systems to different inputs. These assertions may be made simpler with the use of reduction equations, which facilitates control system design and analysis. Reduction equations are helpful in signal processing to address problems related to spectrum analysis, convolution, and filtering. They provide efficient computation of filter responses, convolution integrals, and other signal processing operations **[20]**. In communication systems, the $\widetilde{\boldsymbol{H}}$-function may be used to examine channel characteristics, modulation techniques, and signal propagation. Reduction formulas make it easier to handle complex mathematical expressions, which enhances comprehension and communication system design.

1.6 Fluid Dynamics: Reduction formulas can help simplify integrals involving $\widetilde{\boldsymbol{H}}$-function that appear in the analysis of fluid flow and heat conduction. The $\widetilde{\boldsymbol{H}}$-function may come up in a discussion of viscous flow. Reduction equations can be used to simplify integrals related to the velocity and pressure fields in viscous flow scenarios. This simplification is necessary to get analytical results and comprehend the dynamics of the flow **[21]**. In the context of conduction, convection, or radiation, the $\widetilde{\boldsymbol{H}}$-function may be seen in the solutions to heat conduction problems. Reduction formulas are helpful resources that make it easier to investigate heat transport mechanisms effectively by making integrals pertaining to temperature distributions simpler **[22]**. The $\widetilde{\boldsymbol{H}}$-function is often used in fluid dynamics to solve boundary value problems, especially when viscosity and thermal conductivity are taken into consideration. The use of reduction formulas is crucial when handling mathematical issues.

1.7 Finance and Economics: The $\widetilde{\boldsymbol{H}}$-function can also be found in financial mathematics and economics, where it is used in option pricing models, risk assessment, and modeling stochastic processes. Reduction formulas can aid in evaluating complex financial derivatives and risk measures **[23].**The $\widetilde{\boldsymbol{H}}$-function is a versatile mathematical tool that may be used to assess risk and characterize various financial processes. Reduction equations are used to handle complex computations related to stochastic process modeling, risk management, and option pricing. One helpful tool that may be applied to option pricing models is the $\widetilde{\boldsymbol{H}}$-function. It is present in the partial differential equation solutions that represent the dynamics of financial derivatives [14]. Stochastic processes are often used in financial modeling to capture the ambiguity and unpredictability inherent in financial markets. The $\widetilde{\boldsymbol{H}}$-function may be involved in stochastic differential equations (SDEs). These answers are made simpler by reduction equations, which make it easier to comprehend the dynamics of financial markets.

1.8 Special Function Identities: Reduction formulas contribute to the collection of special function identities and relationships, which are essential tools for mathematicians and researchers working with various types of special functions [18].Overall, the reduction formula of the $\widetilde{\boldsymbol{H}}$-function serves as a versatile tool for simplifying and solving problems in various fields, where complex integrals or equations involving the $\widetilde{\boldsymbol{H}}$-function need to be handled analytically. Its applications extend to both theoretical investigations and practical computations, making it an important mathematical concept in diverse scientific disciplines [27]. Reduction formulas provide mathematicians with powerful tools for theoretical inquiry, contributing significantly to the collection of special function identities. These formulas improve mathematical theory by aiding in the understanding of the many linkages and features found in special functions.

II. REDUCTION FORMULAE FOR THE $\bar{H}$- FUNCTION FORMULA:

This function will be defined and represented in the following manner have been given by Buschman and Srivastava

$$\widetilde{\mathrm{H}}_{P,Q}^{M,N}[z] = \widetilde{\mathrm{H}}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{\phi}(\xi)z^{\xi}\,d\xi, \quad (3)$$

Where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(b_j-\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \quad (4)$$

This contains fractional powers of some of the gamma function. Here, and throughout the paper a_j ($j = I,..., P$) and bj (j = I ,..., Q) are complex numbers, $\alpha_j \geq 0\ (j=1,\dots,P), \beta_j \geq 0\ (j=1,\dots,Q)$(not all zero simultaneously) and the exponents A_j (j = 1,...,N) and B_j (j = M +1,...,Q) can take on non- integer values which we assume to be positive for standardization purposes. The contour in (3) is imaginary axis Re (ξ) =0. Evidently, when the exponents

A_j (j = 1,..., N) and B_j (j = M+1,...,Q) are all equal to unity, the $\widetilde{\boldsymbol{H}}$-function reduces to the well-known Fox's H – function. We have

$$\widetilde{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] = H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]$$

Where the function on the R.H.S. is Fox H- function

The following sufficient conditions for the absolute convergence of the defining integral for $\widetilde{\boldsymbol{H}}$-function given by Eq. (3) have been given by Buschman and Srivastava

$$\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}A_j\alpha_j - \sum_{j=M+1}^{Q}B_j\beta_j - \sum_{j=N+1}^{P}\alpha_j > 0 \quad (5)$$

And $|arg(z)| < \frac{1}{2}\pi\Omega,$ (6)

Where $\boldsymbol{\Omega}$ is given by (5).

The following behaviour of the $\widetilde{\boldsymbol{H}}$-function for small and large values of z as recorded by Saxena and Gupta will be required in the sequel.

$\bar{H}_{P,Q}^{M,N}[z] = O(|z|^g)$ for small z ,Where g = $\min_{1\leq j\leq M}[Re(b_j/\beta_j)]$ and $\bar{H}_{P,Q}^{M,q}[z] = O(|z|^h)$ for large z,

Where h = $\max_{1\leq j\leq N}$ Re$[A_j\{(a_j-1)/\alpha_j\}]$ and the conditions given by (5) and (6) are also satisfied.

The function [z] occurring in the paper stands for the well-known Fox H-function, defined and represented in the following manner

$$H[z] = H_{p,q}^{m,n}\left[x\left|\begin{matrix}(c_j,y_j)_{1,q}\\(d_j,\delta_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s)z^s ds \quad (6)$$

Where $i=\sqrt{-1}$, and $\theta(s) =$

$$\frac{\prod_{j=1}^{m}\Gamma(d_j-\delta_j s\ \prod_{j=1}^{n}\Gamma(1-c_j+y_j s)}{\prod_{j=m+1}^{q}\Gamma(1-d_j+\delta_j s)\prod_{j=n+1}^{p}\Gamma(c_j-y_j s)} \quad (7)$$

The nature of contour L, the conditions of convergence of integral (6)

FORMULA I:

$$\widetilde{\mathrm{H}}_{p+r,q+2r}^{m+2r,n}\left[\mathrm{Z}\left|\begin{matrix}{}_1(a_j,A_j;\alpha_j)_n,\ {}_{n+1}(a_j,A_j)_p,\ {}_1(d_i,D_i)_r\\{}_1(d_i+1,D_i)_r,\ {}_1(k_id_i,k_iD_i)_r,\ {}_1(b_j,B_j)_m,\ {}_{m+1}(b_j,B_j;\beta_j)_q\end{matrix}\right.\right]$$

$$=\left(\prod_{i=1}^{r}k_i\right)^{-1}\widetilde{\mathrm{H}}_{p,q+r}^{m+r,n}\left[Z\left|\begin{matrix}{}_1(a_j,A_j;\alpha_j)_n,\ {}_{n+1}(a_j,A_j)_p\\{}_1(k_id_i+1,k_iD_i)_r,\ {}_1(b_j,B_j)_m,\ {}_{m+1}(b_j,B_j;\beta_j)_q\end{matrix}\right.\right] \quad (8)$$

There is a proof for equation (8), and it is comparable to the proof for equation (8). Here is a broad summary.k_i>0 for i=1, 2, 3… r this condition specifies that each $k_i$is greater than zero for the given indices i.

Proof: To illustrate (8) using an equation

$$\text{LHS} = \frac{1}{2\pi i}\int_L \frac{\prod_{i=1}^{r}\Gamma(d_i+1-D_i s)\prod_{i=1}^{r}\Gamma(k_i d_i-k_i D_i s)}{\prod_{i=1}^{r}\Gamma(d_i-D_i s)}\,\theta(s)\, z^s \, ds$$

$$= \prod_{i=1}^{r}(d_i - D_i s)\,\frac{1}{2\pi i}\prod_{i=1}^{r}\Gamma(k_i d_i - k_i D_i s)\theta(s) z^s\, ds$$

$$= \frac{1}{2\pi i}\prod_{i=1}^{r}\frac{1}{k_i}(k_i d_i - k_i D_i s)\prod_{i=1}^{r}\Gamma(k_i d_i - k_i D_i s)\theta(s) z^s ds$$

$$= \frac{1}{2\pi i}\prod_{i=1}^{r}(k_i)^{-1}\frac{\Gamma(k_t d_i + 1 - k_i D_i s)}{\prod_{i=1}^{r}\Gamma(k_t d_i + 1 - k_i D_i s)}\prod_{i=1}^{r}(k_i d_i - k_i D_i s)\theta(s) z^s ds$$

$$= \frac{1}{2\pi i}\prod_{i=1}^{r}(k_i)^{-1}\frac{\prod_{i=1}^{r}\Gamma(k_t d_i + 1 - k_i D_i s)}{\prod_{i=1}^{r}\Gamma(k_i d_i - k_i D_i s)}\prod_{i=1}^{r}\Gamma(k_i d_i - k_i D_i s)\,\theta(s) z^s ds$$

$$= \left(\prod_{i=1}^{r} k_i\right)^{-1}\frac{1}{2\pi i}\int_L \Gamma(k_i d_i + 1 - k_i D_i s)\theta(s) z^s ds$$

$$= \left(\prod_{i=1}^{r} k_i\right)^{-1}\tilde{H}^{m+r,n}_{p,q+r}\left[Z\,\middle|\,\begin{matrix} {}_1(a_j,A_j;\alpha_j)_n,\ {}_{n+1}(a_j,A_j)_p \\ {}_1(k_i d_i+1,k_i D_i)_r\ {}_1(b_j B_j)_m,\ {}_{m+1}(b_j,B_j;\beta_j)_q \end{matrix}\right]$$

Substitute $\alpha_j = 1$, for j = 1,2,………,n and $\beta_j = 1$, for j = m+1,……,q in (8) substitute gets the formula for reducing the Fox H-function as:

$$\tilde{H}^{m+2r,n}_{p+r,q+2r}\left[Z\,\middle|\,\begin{matrix} {}_1(a_j,A_j)_p,\ {}_1(d_i,D_i)_r \\ {}_1(d_i+1,D_i)_r,\ {}_1(k_i d_i,k_i D_i)_r,\ {}_1(b_j,B_j)_q \end{matrix}\right]$$

$$= \left(\prod_{i=1}^{r} k_i\right)^{-1}\tilde{H}^{m+r,n}_{p,q+r}\left[Z\,\middle|\,\begin{matrix} {}_1(a_j,A_j)_p \\ {}_1(k_i d_i+1,k_i D_i)_r,\ {}_1(b_j,B_j)_q \end{matrix}\right] \quad (9)$$

Special cases:

Specializing the parameters in equation (8) and r = 1, in equation (10) simplifies to

$$\frac{\sqrt{\pi}}{2}\sin\phi\sum_{r=0}^{n}\frac{(\cos\phi-1)^r}{r!\,(n-r)!\,2^r}\tilde{H}^{k+1,l}_{p+1,q+1}\left[\frac{z}{\sin^2\phi}\,\middle|\,\begin{matrix} {}_1(a_j,A_j;\alpha_j)_{l,l+1}\ (a_j,A_j)_p\left(\frac{3}{2}+r,1\right) \\ (2+n+r,2),\ {}_1(b_j,B_j)_k,\ {}_{k+1}(b_j,B_j;\beta_j)_q \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty}\frac{(n+r)!}{n!r!}\sin(n+2r+1)\phi\,\tilde{H}^{k+1,l+1}_{p+2,q+2}\left[Z\,\middle|\,\begin{matrix} (1-r,1;1),\ {}_1(a_j,A_j;\alpha_j)_l,\ {}_{l+1}(a_j,A_j)_p,(2+n+r,1) \\ (2+n,2),\ {}_1(b_j,B_j)_k,\ {}_{k+1}(b_j,B_j;\beta_j)_q \end{matrix}\right], \quad (10)$$

$$\tilde{H}^{m+2,n}_{p+1,q+2}\left[\begin{matrix} {}_1(a_j,A_j;\alpha_j)_n,\ {}_{n+1}(a_j,A_j)_p,(d,D) \\ (d+1,D),\ (kd,kD),\ {}_1(b_j,B_j)_m\ ,{}_{m+1}(b_j,B_j;\beta_j)_q \end{matrix}\right]$$

$$= k^{-1}\tilde{H}^{m+1,n}_{p,q+1}\left[Z\,\middle|\,\begin{matrix} {}_1(a_j,A_j;\alpha_j)_n,\ {}_{n+1}(a_j,A_j)_p \\ (kd+1,kD),\ {}_1(b_j,B_j)_m,\ {}_{m+1}(b_j,B_j;\beta_j)_q \end{matrix}\right] \quad (11)$$

There is a proof for equation (11) result given by Devra, and it is comparable to the proof for equation (10) result given by Cook. Here is a broad summary. $k_i > 0$ for i=1, 2, 3… r this condition specifies that each k_i is greater than zero for the given indices i.

FORMULA II:

$$\tilde{H}^{m+r,n+r}_{p+2r,q+r}\left[Z\,\middle|\,\begin{matrix} {}_1(1+k_i c_i,k_i C_i;1)_r,\ {}_1(a_j,A_j;\alpha_j)_n,\ {}_{n+1}(a_j,A_j)_p,\ {}_1(c_i,C_i)_r \\ {}_1(c_i+1,C_i)_r,\ {}_1(b_j,B_j)_m,\ {}_{m+1}(b_j,B_j;\beta_j)_q \end{matrix}\right]$$

$$= [\prod_{i=1}^{r}(-1)^{-1} k_i]^{-1} \overline{H}^{m,n+r}_{p+r,q}\left[Z \middle| \begin{matrix} {}_1(k_i c_i, k_i C_i 1)_r, {}_1(a_j, A_j; \alpha_j)_n, {}_{n+1}(a_j, A_j)_p \\ {}_1(b_j, B_j)_1, {}_{m+1}(b_j, B_j; \beta_j)_q \end{matrix} \right] \quad (12)$$

Equation (12) with k>0 this equation, when k>0, Equation (12) with r=1.

Special cases:

A reduction formula for the $\overline{H}$-function of Fox by setting a_j=1, for j=1,2,………..,n and β_j=1, for j=m+1,…….,q in Equation (12) , we can proceed as follows:

$$\widetilde{H}^{m+r,n+r}_{p+2r,q+r}\left[Z \middle| \begin{matrix} {}_1(1 + k_i c_i, k_i C_i)_r, {}_1(a_j, A_j)_P, {}_1(c_i, C_i)_r \\ {}_1(c_i + 1, C_i)_r, {}_1(b_j, B_j)_q \end{matrix} \right] =$$

$$[\prod_{i+1}^{r}(-1)^i k_i]^{-1} \overline{H}^{m,n+r}_{p+r,q}\left[Z \middle| \begin{matrix} {}_1(k_i c_i, k_i C_i)_r, {}_1(a_j, A_j)_p \\ {}_1(b_j, B_j)_q \end{matrix} \right] \quad (13)$$

Specializing the parameters in equation (12) and r = 1, in equation (9) simplifies to

$$\widetilde{H}^{m+1,n+1}_{p+2,q+1}\left[Z \middle| \begin{matrix} 1 + kc, kC; 1), {}_1(a_j, A_j; \alpha_j)_{n,} {}_{n+1}(a_j, A_j)_p (c, C) \\ (c + 1, C), {}_1(b_j, B_j)_m, {}_{m+1}(b_j, B_j; \beta_j)_q \end{matrix} \right]$$

$$= (-k)^{-1} \overline{H}^{m,n+1}_{p+1,q}\left[Z \middle| \begin{matrix} (kc, kC; 1), {}_1(a_j, A_j; \alpha_j)_n, {}_{n+1}(a_j, A_j)_p \\ {}_1(b_j, B_j)_m, {}_{m+1}(b_j, B_j; \beta_j)_q \end{matrix} \right] \quad (14)$$

There is a proof for equation (14) result given by Devra, and it is comparable to the proof for equation (8) result given by Cook. Here is a broad summary.k_i>0 for i=1, 2, 3… r this condition specifies that each $k_i$is greater than zero for the given indices i.

III. Application of $\widetilde{H}$-function

Reduction formulae for the generalized hyper geometric function are used in many different areas of mathematics and science. These are a few typical uses-

3.1 Evaluation of Integrals: Reduction formulae are commonly used in the evaluation of integrals when utilizing hyper geometric functions. Complex integrals can be represented in terms of simpler hyper geometric functions using reduction equations, which facilitate integration [15].

3.2 Summation of Series: In the context of infinite series, hyper geometric series are commonly seen. Reduction equations allow one hyper geometric series to be transformed into another, which can make the original series easier to understand and more manageable.

3.3 Particular Functions and Identities: Specific Functions and Identities: Hyper geometric functions are fundamental special functions that arise in the resolution of many differential equations and mathematical difficulties. By helping to create connections between different hyper geometric functions, reduction formulas disclose identities and interconnections.

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3.5 Particular Functions and Identities: Specific Functions and Identities: Hyper geometric functions are fundamental special functions that arise in the resolution of many differential equations and mathematical difficulties [4]. By helping to create connections between different hyper geometric functions, reduction formulas disclose identities and interconnections.

3.6 Sequence Summation: Hyper geometric series are frequently seen in the setting of infinite series. It is possible to convert one hyper geometric series into another using reduction equation, which can help the original series, become more manageable and simpler to grasp.

3.7 Specific functions and identities: Functions and identities that are specific Fundamental special functions known as hyper geometric functions are encountered when solving a variety of differential equations and mathematical challenges. Reduction formulas reveal identities and relationships by interconnecting several hyper geometric functions,

*3.8 Important Unique Features and Roles***:** Hyper geometric functions are fundamental special functions that may be used to solve a variety of differential equations and mathematical problems [5]. The discovery and disclosure of identities and connections among different hyper geometric functions are made easier by reduction equations.

3.9 Physical Applications: Hyper geometric functions may be found in a wide range of scientific fields, including quantum physics, electromagnetic theory, and statistical mechanics. Reduction equations are used to express solutions in terms of known functions and simplify them.

3.10 Number Theory: Reduction equations are occasionally used in the investigation of anomalous hyper geometric function values, which can have applications in number theory. These connections are often studied in the context of modular forms and elliptic functions.

3.11 Neural Networks and Deep Learning: Neural networks in particular are AI models that mainly depend on optimization. Improved performance in high-dimensional environments and faster convergence of training methods are made possible by the $\tilde{\boldsymbol{H}}$-function capacity to simplify complex loss functions.

3.12 Reinforcement Learning: Reinforcement learning models in robotics and autonomous systems rely on addressing stochastic processes, which frequently call for integration across complicated functions. Value functions and optimal policies can be efficiently computed with the aid of reduction equations for generalized functions.

3.13 Surgical Robots: Reduction formulas are used to simplify the mathematical models that control motion, precision, and stability in robotics. For instance, intricate control systems are needed for robotically assisted surgeries in order to maneuver through sensitive settings like human tissue, where even minor mistakes can have serious repercussions. The algorithms controlling these robots make use of the $\tilde{H}$-function to guarantee precise and secure operations.

3.14 Wearable AI for Rehabilitation: Artificial intelligence systems that anticipate and adjust to user demands can be useful in robotic prosthetics and rehabilitation equipment. Reduction formulas can help with real-time support and movement optimization, making sure these devices give the support that's required.

IV.CONCLUSION

Reduction formulas are mathematical identities used to reduce a complicated expression into a simpler form. They are particularly helpful when dealing with integrals, trigonometric functions, and other complex mathematical operations. By using these formulas, we can transform a problem into a more manageable form, making it easier to solve.These formulas are just a starting point, and there are many more reduction formulas used in various mathematical contexts. They play a crucial role in simplifying problems and are valuable tools for mathematicians, physicists, and engineers in solving complex equations and integrals. When working with reduction formulas, it's important to remember the basic principles of integration and differentiation. The broad importance and relevance of reduction formulae in mathematics. It stresses their usefulness in reducing difficult statements and making problem-solving easier, particularly in the context of integrals, trigonometric functions, and other mathematical procedures. It rightly emphasizes that reduction formulae are not restricted to a single area but may be found in a variety of mathematical situations. Reduction formulae for the generalized hyper geometric function are crucial tools in the field of mathematical analysis. We obtain the capacity to explore vast mathematical landscapes by using these formulae, which convert convoluted statements into more understandable ones. This not only simplifies the solving of difficult problems, but also reveals linkages between various hyper geometric functions. a profound mathematical beauty that spreads its effect across multiple areas, from integral calculus to differential equations and beyond.

V. FUTURE SCOPE

The future scope of generalized hyper geometric function reduction formulae offers promise for numerous fields of mathematical research and application. Here are some potential future research directions:

- *Computational Advances:* As computational mathematics and symbolic computation tools progress, new reduction formulae may be discovered. High-performance computing and symbolic manipulation methods can help in the discovery of complicated mathematical connections.
- *Applications of Special Functions in Applied Mathematics:* As the use of special functions, including hyper geometric functions, expands in various branches of applied mathematics, the development of reduction formulas can provide efficient solutions to problems in fields such as physics, engineering, and computer science.
- *Connections to Number Theory:* Exploring linkages between hyper geometric functions and number theory may lead to the discovery of novel reduction formulae. Number theorists may discover fascinating correlations that give light on the features of special functions.
- *Representation Theory:* The study of hyper geometric functions in the context of representation theory and algebraic structures may reveal deeper linkages and give insights into the nature of reduction formulae.
- *Generalizations and Extensions:* Generalizations and Extensions: Mathematicians may investigate generalizations or extensions of the hyper geometric function that lead to new reduction formulae. This might include adding new parameters, exploring more complex series, or investigating hyper geometric functions in various mathematical spaces.

- *Educational Tools:* As our understanding of reduction formulae grows, there is potential for the development of educational tools and resources to aid in the teaching and learning of complex mathematical topics. These tools may be especially useful for new students and researchers joining the subject.

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