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Some results on Uniformly Fuzzy N-Bounded Linear Operators

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Abstract— **Here, in the present paper we have introduced the concept of uniformly Fuzzy n- bounded linear operator from a fuzzy n-normed linear space to another fuzzy n- normed linear space and we established some results on uniformly boundedness on fuzzy n-normed linear space.**

Keywords— Uniformly fuzzy n-bounded linear operator

I. INTRODUCTION

Fuzzy set theory is a useful tool to describe a situation in which the data is imprecise or vague or there is no clear cut boundary. Fuzzy set handle such situation by attributing a degree of membership to which a certain object belongs to the set. In 1965, Fuzzy set was introduced by Zadeh as follows [1-5]

Let X be a set, a fuzzy set A in X is characterised by a membership function

$\mu_A : X \to [0, 1]$

Later, in 1968 Chang introduced fuzzy topology as a family τ of fuzzy sets in X which satisfies the following conditions

(i) $\emptyset, X \in \tau$

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$

(iii) If $A_i \in \tau$ for each $i \in \Lambda$ then $\cup A_i \in \tau$

In 1976, Lowen modified this definition as all constant functions should belong to τ otherwise constant functions will not be continuous [6,7].

A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler in [4]. Katsaras [6] in 1984 first introduced the notion of fuzzy norm on a linear space.

In 1992, Felbin [3] introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding metric associated this fuzzy norm is of Kaleva type [5] fuzzy metric. She also introduced an idea of fuzzy bounded linear operator, the norm of which is a fuzzy number [8,9]. In 2003 Xiao and Zhu [11] redefined, in a more general setting the idea of Felbin's [3] definitions of fuzzy norm of a linear operator from a normed linear space to another fuzzy normed linear space [10].

In this paper, we introduce the concept of uniformly fuzzy n-bounded linear operator on a fuzzy n-normed linear space to another fuzzy n-normed linear space and some results on uniformly fuzzy n-bounded linear operators are established [11].

T. Bag, S.K. Samanta [2] have proved some results on uniformly fuzzy boundedness of fuzzy linear operator on a fuzzy normed linear space using fuzzy norm, we have generalized this concept in a fuzzy n-normed linear space [12].

II. PRELIMANARIES

Definition 2.1. Let X be a real vector space of dimension greater than 1 and let $\|.,. . . . .\|$ be a real valued function on X x ……xX satisfying the following conditions

(1) $\|x_1,x_2,\ldots,x_n\| = 0$ if and only if $x_1,x_2,\ldots,x_n$ are linearly dependent

(2) $\|x_1,x_2,\ldots,x_n\|$ is invariant under any permutation

(3) $\|x_1,x_2,\ldots,\alpha x_n\| = |\alpha|\ \|x_1,x_2,\ldots,x_n\|$ for any $\alpha \in R$

(4) $\|x_1,x_2,\ldots,x_{n-1},y+z\| \le \|x_1,x_2,\ldots,x_{n-1},y\| + \|x_1,x_2,\ldots,x_{n-1},z\|$

$\|.,.,.,.,.,.\|$ is called n-norm on X and the pair $(X, \|.,.,.,.,.\|)$ is called a n-normed linear space.

Definition 2.2. Let X be a linear space over a field F. A fuzzy subset N of X x X x ….X x R (R is the set of real numbers) is called a fuzzy n-norm on X if and only if

(N_1) for all $t \in R$, with $t \le 0$, $N(x_1,x_2,\ldots,x_n,t) = 0$

(N_2) for all $t \in R$, with $t>0$, $N(x_1,x_2,\ldots,x_n,t) = 1$ if and only if $x_1, x_2, \ldots x_n$ are linearly dependent.

(N_3) $N(x_1,x_2,\ldots,x_n,t)$ is invariant under any permutation of $x_1, x_2, \ldots x_n$.

(N_4) for all $t \in R$, with $t>0$

$$N(x_1, x_2,\ldots,cx_n, t) = N(x_1, x_2,\ldots,x_n, \frac{t}{|c|}) \text{ if } c\neq 0,\ c\in F.$$

(N_5) for all $s, t \in R$, $N(x_1,x_2,\ldots,x_n+x'_n, s+t) \ge \min\{N(x_1,x_2,\ldots,x_n,s), N(x_1, x'_2,\ldots,x'_n,t)\}$

(N_6) $N(x_1,x_2,\ldots,x_n,\bullet)$ is non-decreasing function of R and $\lim_{t\to\infty} N(x_1, x_2,\ldots, x_n, t) = 1$

Then (X, N) is called a fuzzy n-normed linear space.

Example 2.1: Let $(X,\|.,.,.,.,.\|)$ be n-normed linear space define

$$N(x_1,x_2,..,x_n,t) = \frac{t}{t+\|x_1,x_2,....,x_n\|} \quad \text{when } t > 0,$$

$t\in R$, $(x_1,x_2,\ldots,x_n) \in XxXx\ldots.X \quad = \quad 0$

when $t \le 0$, $t\in R$,

Then (X, N) is fuzzy n-normed linear space.

Theorem 2.1 .[7] Let (X, N) be a fuzzy n- normed linear space. Assume that

(N_7) $N(x_1,x_2,..,x_n,t) > 0$

For all $t > 0$ implies $x_1, x_2,\ldots x_n$ are linearly dependent, define $\|x_1,x_2,...,x_n\|_\alpha = \inf\{ t: N(x_1,x_2,\ldots x_n,t) \ge \alpha \in (0,1) \}$. Then $\{\|.,.,.,.,.,.\|_\alpha: \alpha \in [0,1) \}$ is an ascending family of n-norms on X. These n-norms are called α-n-norms on X corresponding to the fuzzy n-norms.

Definition 2.4[8]. A fuzzy n-linear operator T is a function from $X_1xX_2x\ldots xX_n$ to $Y_1xY_2x\ldots.Y_n$ where $X_1.X_2\ldots X_n$ are subspaces of fuzzy n-normed linear space (X, N_1) and $Y_1,Y_2,..Y_n$ are subspaces of fuzzy n-normed linear space (Y, N_2) such that

$T(\sum_{i_n=1}^{n} x_1^{(i_n)},\sum_{i_{n-1}=1}^{n} x_2^{(i_{n-1})},\sum_{i_{n-2}=1}^{n} x_3^{(i_{n-2})},\ldots.. \sum_{i_2=1}^{n} x_{n-1}^{(i_2)},\sum_{i_1=1}^{n} x_n^{(i_1)}) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}\ldots..\sum_{i_n=1}^{n} T (x_1^{(i_n)}, x_2^{(i_{n-1})}, x_3^{(i_{n-2})}, \ldots\ldots x_{n-1}^{(i_2)}, x_n^{(i_1)})$

and $T(\alpha_1x_1,\alpha_2x_2,\ldots.,\alpha_nx_n) = \alpha_1\alpha_2...\alpha_n T(x_1,x_2,\ldots,x_n)$, where $\alpha_1,\ldots\alpha_n \in[0,1]$.

Definition 2.5. Let T be a fuzzy n-linear map from $X_1xX_2x\ldots xX_n$ to $Y_1xY_2x\ldots xY_n$ where $X_1,\ldots X_n$ are subspaces of (X, N_1) and $Y_1,\ldots Y_2$ are subspaces of (Y, N_2) then it is said to be strongly fuzzy n-bounded on $X_1xX_2x\ldots xX_n$ if and only if $\exists$ a positive real number M such that $\forall (x^1,\ldots,x^n) \in X_1x\ldots xX_n$ and $\forall s\in R$,

$N_2[T(x^1,\ldots..,x^n),s] \ge N_1[(x^1,\ldots.,x^n),\frac{s}{M}]$

Definition 2.6. Let T be a fuzzy n-linear map from $X_1xX_2x\ldots xX_n$ to $Y_1xY_2x\ldots xY_n$ where $X_1,\ldots X_n$ are subspaces of (X, N_1) and $Y_1,\ldots Y_2$ are subspaces of (Y, N_2) then it is said to be fuzzy n-continuous at $(x_0^1,\ldots,x_0^n) \in X_1x\ldots xX_n$ for given $\varepsilon > 0$, $\alpha\in(0,1)$, $\exists\ \delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon) \in (0,1)$, such that $\forall (x^1,\ldots. ,x^n) \in X_1x\ldots..xX_n$

$N_1 [(x^1,\ldots.., x^n) - (x_0^1,\ldots,x_0^n), \delta] > \beta \Rightarrow N_2 [T(x^1, ..,x^n) - T(x_0^1,\ldots,x_0^n), \varepsilon] > \alpha$

If T is fuzzy n-continuous at each point of $X_1x\ldots xX_n$ then T is fuzzy n-continuous on $X_1x\ldots xX_n$.

Definition 2.7. A fuzzy n-linear map T from $X_1xX_2x..xX_n$ to $Y_1xY_2x..Y_n$ where $X_1,..X_n$ are subspaces of (X,N_1) and $Y_1,\ldots,Y_n$ are subspaces of (Y,N_2) is said to be strongly fuzzy n-continuous at $(x_0^1,\ldots,x_0^n) \in X_1x\ldots X_n$ if for each $\varepsilon > 0$, $\exists\ \delta > 0$ such that $\forall (x^1, \ldots,x^n) \in X_1x\ldots xX_n$

$N_2 [T(x^1,\ldots, x^n) - T(x_0^1,\ldots.,x_0^n), \varepsilon] \ge N_1 [(x^1,\ldots., x^n) - (x_0^1,\ldots,x_0^n), \delta]$

Definition 2.8. A fuzzy n-linear map T from $X_1x\ldots xX_n$ to $Y_1x\ldots xY_n$ where $X_1,..X_n$ are subspaces of (X,N_1) and $Y_1,..,Y_n$ are subspaces of (Y,N_2) is said to be weakly fuzzy n-continuous at $(x_0^1,\ldots,x_0^n) \in X_1x\ldots xX_n$ if for a given $t > 0$, $\alpha\in(0,1)$, $\exists\ \delta = \delta(\alpha,\varepsilon) > 0$ such that $\forall (x^1,\ldots,x^n) \in X_1x\ldots xX_n$

$N_1 [(x^1,\ldots., x^n)-(x_0^1,\ldots.,x_0^n), \delta] \ge \alpha \Rightarrow N_2 [T (x^1,\ldots., x^n)-T(x_0^1,\ldots,x_0^n),\varepsilon] \ge \alpha$

III. UNIFORMLY FUZZY 2-BOUNDED LINEAR OPERATOR

In this section we have defined uniformly boundedness of Fuzzy n-linear operator and some results on uniformly boundedness are established.

Definition 3.1. Let $T : X_1x\ldots xX_n \to Y_1x\ldots xY_n$ where $X_1,..,X_n$ are subspaces of (X,N_1) and $Y_1,\ldots,Y_n$ are subspaces of (Y,N_2) then T is said to be uniformly fuzzy n-bounded linear operator if $\exists\ M > 0$ such that $\|T(x^1, ..., x^n)\|_\alpha^2 \le M\ \|(x^1, ..., x^n)\|_\alpha^1$, $\forall\ \alpha \in (0,1)$ where $\|.,.,...,\|_\alpha^1$ and $\|.,.,.,.,.,.\|_\alpha^2$ are α-n-norms of N_1 and N_2 respectively.

Theorem 3.1. Let (X,N_1) and (Y,N_2) be two fuzzy n-normed linear spaces satisfying (N_6) and (N_7). Let $T : X_1x..xX_n \to Y_1x..xY_n$ be a n-linear operator. Then T is strongly fuzzy n-bounded iff it is uniformly fuzzy n-bounded with respect to α-n norms of N_1 and N_2 .

Proof. Let $\|.,.,.,.,.\|_\alpha^1$ and $\|.,.,.,.,.\|_\alpha^2$ are α-n norms of N_1 and N_2 respectively. First we suppose that T is strongly fuzzy n-bounded thus $\exists\ M > 0$ such that

$N_2[T(x^1,\ldots, x^n),s] > N_1[(x^1,\ldots, x^n),\frac{s}{M}] \quad \forall (x^1,\ldots,x^n) \in X_1x\ldots xX_n$,$s \in R$.

i.e. $N_2[T(x^1, \ldots,x^n),s] > N_1[M(x^1,\ldots,x^n),s]$ ……..(1)

Now

$\|M(x^1, ..., x^n)\|_\alpha^1 < t \Rightarrow \wedge \{ s: N_1(M(x^1,\ldots, x^n),s) \ge \alpha\} < t$

$\Rightarrow \exists\ s_0 < t$ such that $N_1(M(x^1,\ldots, x^n),s_0) \ge \alpha$

$\Rightarrow \exists\ s_0 < t$ such that $N_2(T(x^1,\ldots, x^n),s_0) \ge \alpha$ …by (1)

$\Rightarrow \|T(x^1, ..., x^n)\|_\alpha^2 \ge s_0 < t$

Hence $\|T(x^1,..,x^n)\|_\alpha^2 \le M\ \|(x^1,..,x^n)\|_\alpha^1 \quad \forall\ \alpha \in (0,1)$

This implies that T is uniformly fuzzy n-bounded with respect to α-n norms, $\forall\ \alpha \in (0,1)$.

Conversely, suppose that, there exists $M > 0$ such that

$\|T(x^1, ..., x^n)\|_\alpha^2 \le M\|(x^1,..,x^n)\|_\alpha^1$…………… (2)

Holds $\forall\ \alpha \in (0,1)$ and $\forall (x^1,..,x^n) \in X_1x\ldots xX_n$

$r < N_1[M(x^1,..,x^n), s] \Rightarrow r < \wedge\{ \alpha \in (0,1) : \|M(x^1, ..., x^n)\|_\alpha^1 \le s\}$

$\Rightarrow \exists\ \alpha_0 \in (0,1)$ such that $r < \alpha_0$ and $\|M(x^1, ..., x^n)\|_\alpha^1 \le s$

$\Rightarrow \|T(x^1,..,x^n)\|_\alpha^2 \le s$ by (2)

$\Rightarrow N_2[T(x^1,..,x^n),s] \ge \alpha_0 > r$

Hence $N_2[T(x^1,..,x^n),s] \ge N_1[M(x^1,...,x^n),s] = N_1[(x^1,...,x^n),\frac{s}{M^n}]$

i.e. T is strongly strongly fuzzy n-bounded and hence theorem is follows.

Theorem3.2. Let (X, N_1) and (Y,N_2) be two fuzzy n-normed linear spaces, T is a linear operator from $X_1x\ldots X_n$ to $Y_1x\ldots xY_n$ then T is strongly fuzzy n-continuous every on $X_1x\ldots xX_n$ if is strongly fuzzy n-continuous at a point $x_0^1,\ldots,x_0^n \in X_1x\ldots X_n$.

Proof. Since T is strongly fuzzy n- continuous at $x_0^1,\ldots,x_0^n \in X_1x\ldots X_n$.

, thus for each ε > 0, ∃ δ > 0 such that ∀ $(x^1,\ldots x^n) \in X_1x\ldots X_n$.

We have,$N_2[T(x^1,\ldots x^n)–T(x_0^1,\ldots,x_0^n), \varepsilon]\geq$
$N_1[(x^1,\ldots,x^n)-(x_0^1,\ldots,x_0^n), \delta]$

Taking any $(y^1,\ldots,y^n) \in X_1x\ldots xX_n$ and replacing $(x^1,\ldots,x^n)$ by $(x^1,\ldots,x^n)+(x_0^1,\ldots,x_0^n)-(y^1,\ldots,y^n)$, we get
$N_2[T((x^1,\ldots,x^n)+(x_0^1,\ldots,x_0^n)-(y^1\ldots y^n))-T(x_0^1,\ldots x_0^n), \varepsilon]$
$\geq N_1[((x^1,\ldots,x^n)+(x_0^1,\ldots,x_0^n)-(y^1,\ldots,y^n))-(x_0^1,\ldots x_0^n), \delta]$
$\Rightarrow N_2[T(x^1,\ldots,x^n)-T(y^1,\ldots,y^n), \varepsilon]\geq N_1[((x^1,\ldots,x^n)-(y^1,\ldots,y^n)), \delta]$

Since $(y^1,\ldots,y^n)$ is arbitrary, it follows that T is strongly fuzzy n-continuous on $X_1x\ldots Xn$.

Theorem3.3. Let (X,N_1) and (Y,N_2) be two fuzzy n-normed linear spaces, T is a linear operator from $X_1xX_2\ldots xX_n$ to $Y_1xY_2x\ldots xY_n$, then T is weakly fuzzy n-continuous every on $X_1xX_2x\ldots xX_n$ if it is weakly fuzzy n-continuous at a point $(x_0^1,\ldots,x_0^n) \in X_1x\ldots xX_n$.

Proof. Since T is weakly fuzzy n-continuous at $(x_0^1,\ldots,x_0^n)\in$, $X_1x\ldots xX_n$ thus for each ε > 0 and α ∈ (0,1), ∃ δ(α, ε) > 0 such that ∀ $(x^1,\ldots,x^n) \in X_1x\ldots xX_n$. We have

$N_1[(x^1,\ldots,x^n)-(x_0^1,\ldots,x_0^n), \delta] \geq \alpha \Rightarrow N_2[T(x^1,\ldots,x^n)-T(x_0^1,\ldots,x_0^n), \varepsilon] \geq \alpha$

Taking any $(y^1,\ldots,y^n) \in X_1x\ldots xX_n$ and replacing $(x^1,\ldots,x^n)$ by $(x^1,\ldots,x^n)+(x_0^1,\ldots,x_0^n)-(y^1,\ldots,y^n)$, we get

$N_1[((x^1,\ldots,x^n)+(x_0^1,\ldots,x_0^n)-(y^1,\ldots,y^n))-(x_0^1,\ldots,x_0^n), \delta] \geq \alpha$
$\Rightarrow N_2[T((x^1,\ldots,x^n)+(x_0^1,\ldots,x_0^n)-(y^1,\ldots,y^n))-T(x_0^1,\ldots,x_0^n), \varepsilon] \geq \alpha$
$\Rightarrow N_1[(x^1,\ldots,x^n)-(y^1,\ldots,y^n), \delta] \geq \alpha \Rightarrow N_2[T(x^1,\ldots,x^n)-T(y^1,\ldots,y^n), \varepsilon] \geq \alpha$

Since $(y^1,\ldots,y^n)$ is arbitrary, it follows that T is weakly fuzzy n-continuous on $X_1x\ldots xX_n$.

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